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Intrinsic Stochasticity with Many Degrees of Freedom

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We study a simple model equation describing a system with an infinity of degrees of freedom which displays an intrinsically chaotic behavior. Some concepts of fully developed turbulence are discussed in relation to this model. We also develop an approach based on Lyapunov exponent measurements. Numerical results on the distribution of Lyapunov numbers and the power spectrum of the associated Lyapunov vectors are presented and briefly discussed.

KEY WORDS: Intrinsic stochasticity; turbulent fluctuations; energy transfer; Lyapunov exponents.

With current computer technology, it is still difficult to model three-dimensional flows at large Reynolds numbers and to find their statistical properties with sufficient accuracy. A possible way of circumventing these difficulties is to use model equations displaying spatiotemporal chaos, but which are simple enough to make detailed numerical investigation possible. A good candidate for this modeling is the equation[1]

$$\varphi_t + \varphi\varphi_x + \varphi_{xx} + \varphi_{xxxx} = 0 \tag{1}$$

where $\varphi(x, t)$ is a smooth function of space (x) and time (t) and where φ_u denotes $\partial\varphi/\partial u$. In the dimensionless form (1), the external parameter, which could be seen as a kind of Reynolds number, is the length L of the support of φ. The boundary conditions can be chosen as $\varphi = \varphi_x = 0$ at $x = 0, L$ or one may impose L periodicity: $\varphi(x + L) = \varphi(x)$. Both choices lead to a very similar bulk behavior when L is large enough, say, larger than 20. As L

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exceeds some critical value (~ 10) the space-time fluctuations of φ spontaneously reach a turbulent regime, without any external random stirring and for a large class of initial conditions. Obviously equation (1) has only a loose connection with three-dimensional Navier–Stokes equations. However, it remains interesting to use concepts introduced in the study of fully developed turbulence such as energy spectra, transfer of energy, etc. On the other hand, it is possible to consider that Eq. (1) defines a dynamical system with many degrees of freedom and to study its properties from this point of view. In this paper we try to combine these different approaches to get some insight in the dynamical mechanism at work in fully developed turbulence.

The chaotic regime of Eq. (1) defines a statistical ensemble $\{\varphi(x, t)\}$ translationally invariant with respect to t and x, at least for space fluctuations much smaller than L. As usual, observables are in fact obtained through time averages $(\overline{\cdots})$. One of the quantities of primary interest is $S(k) = (|\varphi_k|^2)$ which, in the limit of large L restoring translational invariance, is the Fourier transform of the two-point spatial correlation function. Figure 1 displays $S(k)$ as a function of k using log–log and lin–log coordinates. $S(k)$ is flat near $k = 0$ (Fig. 1a), a property which is reminiscent of the equipartition of energy.[2] It has a sharp maximum near $k_0 = 1/\sqrt{2}$ which is the wave vector of the most unstable fluctuations near the uniform steady state $\varphi = 0$. The spectrum then decreases for $k > k_0$, following first a power law approximately as k^{-4} (Fig. 1a) and then an exponential decay as $\exp(-k/k_1)$ when k becomes larger and larger (Fig. 1b). This behavior is reminiscent of what occurs in three-dimensional fully developed turbulence. This analogy has motivated the following interpretation.

On general grounds, the k^{-4} subrange can be understood as due to discontinuities of the derivative φ_x (that is, cusp points for φ), although this is not obvious at all from the instantaneous shape of the function $\varphi(x, t)$. Indeed one has in mind the phenomenon occurring, e.g., in Burger's equation

$$\partial_t u + u\,\partial_x u = \nu\,\partial_{xx} u$$

In the inviscid case ($\nu = 0$) shocks appear, and for small nonzero viscosity the solution is still reminiscent of these singularities and displays a k^{-2} spectrum for k small enough.

In the same spirit, the exponential decay of $S(k)$ at large k may be related to the existence of singularities of the complex x extension of $\varphi(x, t)$ beyond some fixed distance (of the order of k_1^{-1}) from the real axis.[3] But here we would like to point out a relation between these two parts of the spectrum which derive from the adaptation of an argument first presented by Corrsin in the context of usual (three-dimensional) developed turbulence[4]:

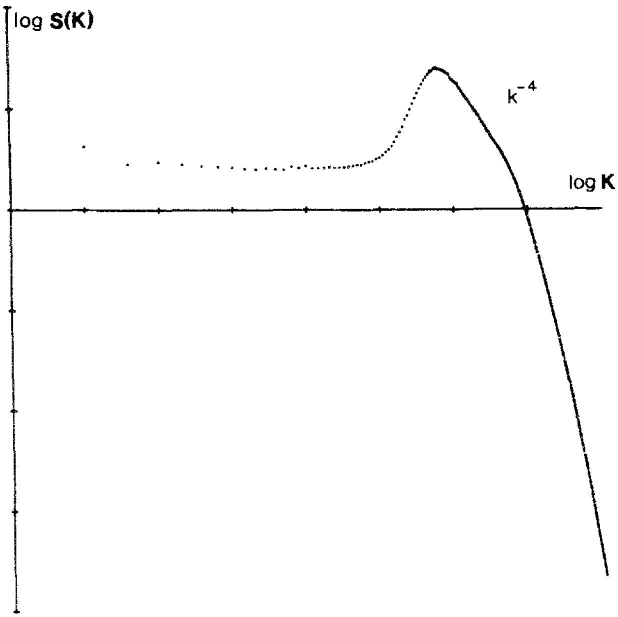

Figure 1a

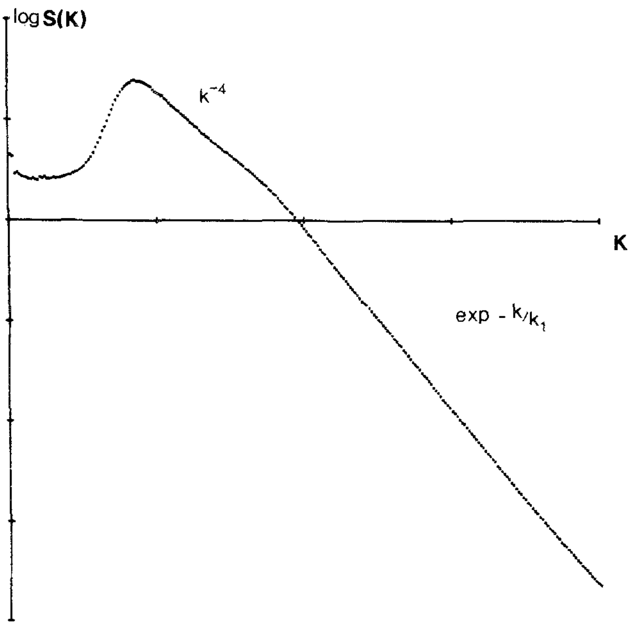

Figure 1b

Fig. 1. Power spectrum $S(k)$ for $L = 512$. (a) Log–log plot: Note the flat spectrum for $k \ll k_0 = 1/\sqrt{2}$, the broad peak at $k \sim k_0$, and the power law decay $S(k) - k^4$ for $k_0 < k < 1$. (b) Lin–log plot: The cross-over from the power law decay to the exponential behavior for $k \gg 1$ is clearly visible. (Courtesy P. Manneville DPh-G/PSRM Saclay).

From (1) and the boundary condition one can derive an "energy budget":

$$\overline{(\varphi_x^2)} = \overline{(\varphi_{xx}^2)} \tag{2}$$

where $\overline{(\varphi_x^2)}$ represents the energy injected by the instability, i.e., the term φ_{xx} in (1), and $\overline{(\varphi_{xx}^2)}$ the dissipation of this energy ensured by the stabilizing term φ_{xxxx} in (1). But whereas more energy is injected than dissipated at k smaller than k_0, dissipation prevails over injection in the large-k range. So the mean energy budget (2) is satisfied only if an energy flux $T(k)$ takes place in the k space from small to large k values. This energy transfer is due to the nonlinear term $\varphi\varphi_x$ in (1). To approach this phenomenon quantitatively, one writes an equation for the energy transfer in k space for large k as

$$dT(k)/dk = -k^4 S(k) \tag{3}$$

where $T(k)$ is the energy flux due to nonlinear terms and $-k^4 S(k)$ the energy dissipated at large k. For Newtonian fluids dissipation comes from the effect of viscosity so that the right-hand side of (3) would read $-\nu k^2 S(k)$. The expression of $T(k)$ as a function of k and $S(k)$ remains to be given. This step involves a nontrivial guess because the local energy transfer (LET) in the k space does not follow unambiguously from the original equations. However we can reason as follows. In the present problem the k^{-4} subrange can be identified with the inertial subrange, that is, a wave vector domain where the evolution is controlled by the nonlinear transfer term since for $k \sim 1$ the growth due to the instability term is nearly compensated for by the decay due to the dissipation term. This compensation implies that $T(k)$ is independent of k in the "inertial" part of the spectrum. If one further assumes that the flux of energy is proportional to the energy density and if one takes into account the k^{-4} power law of the observed spectrum in the inertial subrange one is led to $T(k) \sim k^4 S(k)$. Solving now (3) one finds $S(k) \sim k^{-4} \exp(-k/k_1)$, which would explain the exponential decay of $S(k)$ at large k.

Let us now discuss the long-wavelength part of the spectrum ($0 < k < 0.6k_0$). In this domain $S(k)$ is observed to be almost constant. This is what one would expect from equipartition of energy in a Gibbs ensemble. Actually $S(k)$ is the "energy" of a fluctuation with wave number k. Nevertheless the reason that this should be the energy (as it occurs in the Gibbs–Boltzmann statistical weight) is not entirely clear since it is not the only quantity conserved by the nonlinear term. Indeed the "inviscid" equation:

$$\varphi_t + \varphi\varphi_x = 0 \tag{4}$$

leaves $\int_0^L \varphi^2\,dx$ invariant so that φ^2 may be considered as the energy density, but any quantity of the form $\int_0^L dx\,F\{\varphi\}$, F a smooth function, is also conserved by Eq. (4) and there is no clear reason to choose φ^2 as energy density instead of, say, φ^6 or $\varphi^{2/3}$. In order to clarify this point we have considered the slightly different equation:

$$\varphi_t + \varphi\varphi_x + \varphi_{xx} + \varphi_{xxxx} + \varepsilon\varphi = 0 \tag{5}$$

This equation also has chaotic solutions for sufficiently small positive ε and L large enough, as can be seen from the existence of strictly positive Lyapunov exponents we have computed. The $\varepsilon\varphi$ term ($\varepsilon > 0$) introduces a dissipation at all wavelengths so that in the "inviscid limit" the Riemann invariants are now exponentially damped along the characteristics during the evolution. We have computed the spectrum $S(k)$ of the turbulent fluctuations with Eq. (5). These spectra corresponding to different values of ε are very similar to the spectrum obtained for $\varepsilon = 0$ [Eq. (1)]. In particular the flat part at small k is almost unchanged. This casts doubt on the connection between energy conservation and the flat spectrum at small k. It is also of interest to note that in the "inertial range" the spectrum is of the form k^{-n} with n increasing as ε increases (see Table I). This can be understood by using in a simplified way the idea of LET. In the inertial regime the energy flux in k space obeys the equation

$$dT(k)/dk = 0 \tag{6}$$

but for $\varepsilon > 0$ a k-independent power loss exists. Thus one has to add to the right-hand side of (6) a term $-S(k)$, and if one assumes again $T(k) \sim k^4 S(k)$ one finds

$$d(k^4 S(k))/dk = -\varepsilon S(k)$$

which has the solution

$$S(k) = k^{-4} \exp(\varepsilon/3k^3)$$

so that the spectrum decreases more rapidly than k^{-4} at large k, which we interpret as an increase of the effective exponent. The weakness of this reasoning is of course in the assumed relation between $T(k)$ and $S(k)$.

Table I

eps	0.	0.0005	0.005	0.05
exp	3.99	4.15	4.32	5.15

Up to now the kind of approach used was traced from the standard statistical theory of developed turbulence. In the following we shall join the main flow of more recent approaches to nonlinear phenomena in terms of dynamical systems.

Here we are concerned with a single but functional degree of freedom governed by a partial differential equation. This corresponds to an infinite-dimensional system of discrete degrees of freedom. Extensions to infinite dimensionality of the theory developed for systems of low dimensionality have already been considered from a theoretical point of view (D. Ruelle for Navier–Stokes equations[5]) or numerically (D. Farmer for a differential equation with delay[6]). Analytical results for equation (1) are rather scarce[7] and to get a better knowledge of its turbulent solutions one is left with the recourse to numerical simulations. In this paper we shall report mainly on results concerning the Lyapunov numbers (LN for short) which measure the degree of instability of trajectories in phase space[8] and provide information on the nature of the chaotic behavior of the system.[9]

Here the degrees of freedom are the values of the φ at N space grid points and their evolution can be obtained quite efficiently using a finite difference implicit scheme which requires on the order of N arithmetic operations per time step without stringent numerical stability restrictions.

The LN are associated with the mean unstable directions of motion in phase space and the number of positive LN can be understood as the effective number of "degrees of freedom" of the turbulent system. To make this more precise in the large-L limit we have considered the distribution and values of these LN and their associated "Lyapunov vectors" (LV), and we have tried to reach the mean spatial structure corresponding to the positive LNs. The eigenvalues and eigenvectors defined by the tangent motion fluctuate so that there is some arbitrariness in defining the mean properties of these Lyapunov vectors. The LNs can be determined by measuring how a small parallelepiped in the tangent space to the phase space at P_0 [in our case P_0 is equivalent to the specification of the Cauchy data $\varphi(x)$ at $t=0$] is strained during the evolution of the system.[8] This procedure avoids a brute diagonalization of the matrix of the tangent evolution which would be untractable at large N.

To compute the LNs and the LVs we proceed in several steps:

(i) We start at P_0 in phase space and take an orthogonal family of p arbitrary independent vectors in tangent space: δP_0^i, $i = 1,\dots, p$.

(ii) Then we integrate numerically Eq. (1) during a time step Δt, with initial conditions P_0, $P_0 + \delta P_0^1$, $P_0 + \delta P_0^2 \cdots P_0 + \delta P_0^p$ and determine the vectors $M(\Delta t) \cdot \delta P_0^i = \delta \tilde{P}_0^i$, where $M(\Delta t)$ is the matrix of the tangent motion.

(iii) By standard Gram–Schmidt orthogonalization we define from the

δPs a new family $\delta P_1^i = a_1^i \, \delta\tilde{P}_1^i$ where $a_1^i = |\delta\tilde{P}_1^i|$ and $\delta P_1^j \cdot \delta P_1^i = \delta ij$. At this step we have defined another point in phase space P_1 and a new orthonormal family δP_1^i. An iteration of this procedure allows to define p numbers a_k^i, $i = 1,..., p$ and p orthonormal vectors δP_k^i, $i = 1,..., p$. The average of $\ln(a_k^i)$ for a given i gives the value of the ith LN, while the $\{\delta P^i\}$ gives us what we call the Lyapunov vectors which span in the mean the unstable directions of the system.

It is possible to compute up to 40–50 LN by the process defined above. One of our main result concerns the growth of the number of positive Lyapunov exponents (LN) with the size of the system which is found to be linear with L. Such a result is interesting since the number of positive LN provides estimates of the dimension of strange attractors (which is a measure of the number of degrees of freedom), the entropy, and other quantities which measure the amount of chaos. It has also been established that (i) with the dimensionalization used in writing (1) the largest LN tends to a finite value λ_0 of the order of 0.1 as L increases; (ii) the distribution of positive LN has no obvious singularity near λ_0 from below (Fig. 2). Although our statistics are still rather insufficient. Some recent results[10] confirm this statement. (Fig. 3). This contrasts with the distribution of the growth rates of unstable fluctuations of (1) near $\varphi = 0$. As a function of the wave number k

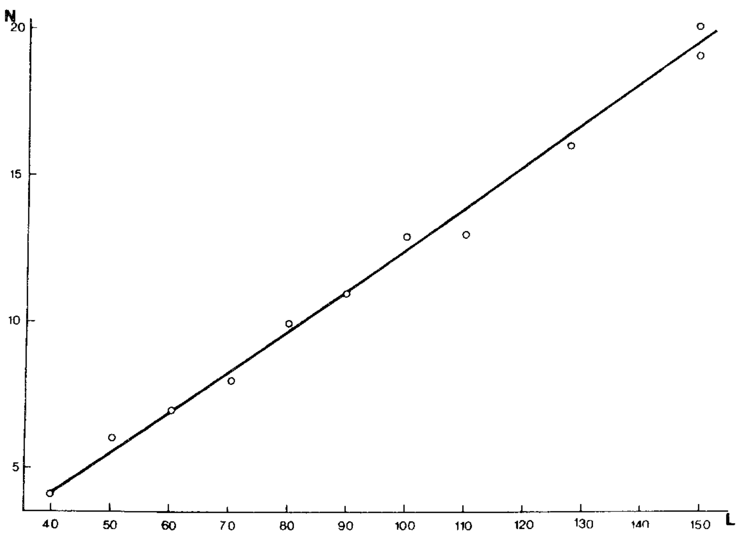

Fig. 2. Number of positive Lyapunov numbers as a function of the length L: $N(L) = 0.14\,L - 1.5$, note (i) that this number reaches zero for $L = 11$ which corresponds fairly well to the value of the onset of turbulence, and (ii) that it grows more slowly than the number of linearly unstable modes ($=L/\pi \sim 0.32\,L$).

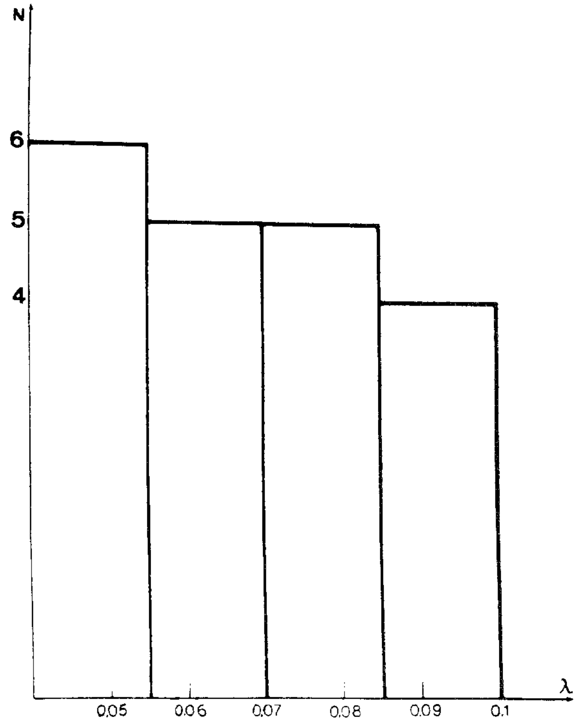

Fig. 3. Distribution of the positive Lyapunov numbers for $L = 400$: There is no evidence of accumulation near the mximum value 0.1 (itself smaller than the growth rate of the most unstable mode).

this growth rate is $\gamma(k) = k^2 - k^4$. For a large L the wave numbers are uniformly distributed with a spacing $2\pi/L$ and the distribution of the positive growth rates has an inverse square root singularity at 0 and at the largest growth rate 0.25.

To get more information about the divergence of trajectories we have computed the mean spectral power of the LV$\{\delta P_k^i\}$. These vectors are actually functions of x and can be Fourier transformed:

$$\delta P_l^i(x) = \sum_k \exp(2i\pi kx/L)\, \pi_l^i(k)$$

For each wave number k and each integer i $(1 < i < p)$ we have computed:

$$\langle |\pi^i(k)|^2 \rangle = (1/M) \sum_{l=1}^{M} |\pi_l^i(k)|^2$$

where M is the number of iterations. The numerical results are given in Fig. 4. The average spectra corresponding to the 15 largest LN were computed using 1280 samples for $L = 128$. These spectra have noticeable features. First of all the large wave number dependence is almost the same for all spectra and the spectral power decays exponentially for $k > 1$. The algebraic part (k^{-4}) of the spectrum of $\varphi(x, t)$ itself is no longer apparent in the spectrum of the LV. The relative spectral power at low wave numbers increases from the first to the 15th vector.

Now we try to get information in the mechanism of energy transfer from the spectra of LV.

Let us suppose that a cascade process takes place in wave vector space. This would imply that energy is transferred from wavevector k_1 to k_2 ($>k_1$) and then to k_3 ($>k_2$). So one of the LV would correspond to the energy transfer from k_1 to k_2 and thus have a peak of spectral power around k_2

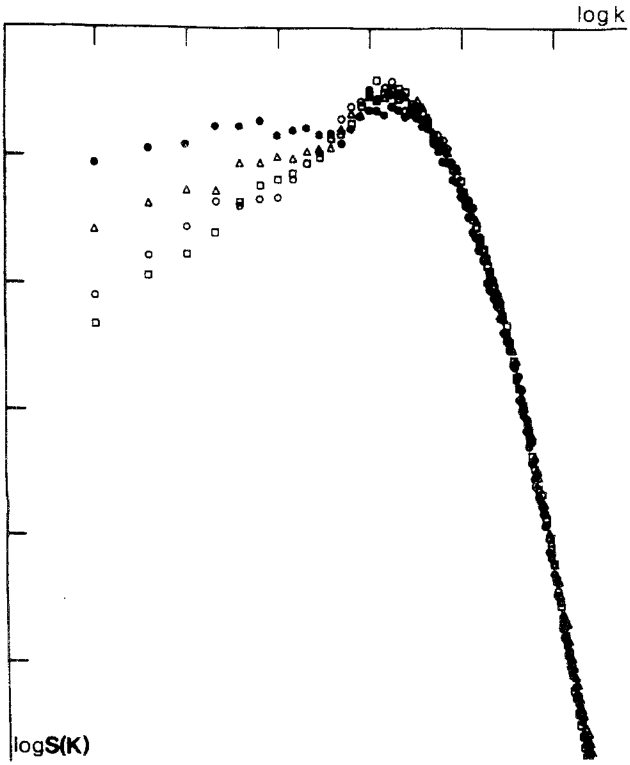

Fig. 4. Power spectra of some "Lyapunov vectors" corresponding to positive Lyapunov Numbers for $L = 128$. The power contents of the long wavelengths steadily increases when passing from the first LN (□) to the 5 (○), then to the 10 (△), and the 15 (●).

because the corresponding dynamics is a growth of the mean spectral power near k_2. The spectra that we have obtained did not display such peaks. A possible reason for this could be the limited width of the inertial region which would inhibit the development of the cascading process.[4b] But another explanation involves the growth and death of localized structures of size $1/k_0$ without any cascading process. Then all the LV share the signature of this structure in the large wave number part of their power spectrum while the small wave number part reflects more or less the effects of various possible arrangements of these localized structures. The energy transfer through the growth and death of these localized structures could be strongly dependent on the number of spatial dimensions. Indeed, one may decompose the tangent motion by solving the spectral problem:

$$\delta\varphi_{xx}^{(\lambda)} + \delta\varphi_{xxxx}^{(\lambda)} + (\varphi\,\delta\varphi^{(\lambda)})_x = \lambda\,\delta\varphi^{(\lambda)} \tag{7}$$

with $\delta\varphi^{(\lambda)} = 0$ for $x = 0$, L, $\varphi(\cdot, \cdot)$ being a "random" solution of (1). This spectral problem has a unusual form. But, as in the localization problem of a quantum particle in a random potential,[11] the random quantity $\varphi(x, t)$ [or $\varphi_x(x, t)$] multiplies of the lower derivatives of $\delta\varphi^{(\lambda)}$, so that φ can still be seen as a random potential. It is well known that the localization depends in a highly nontrivial way on the dimensionality.

The importance of large-scale coherent structures in hydrodynamic turbulence is strongly suggested by experiments[12]; the study of the localization-like problem defined by (7) is a possible starting point for a theoretical approach to this phenomenon. The mechanism proposed above is fairly different from the classical approaches which rest mainly on LET assumptions. Other numerical results are needed to make these ideas more precise.

In conclusion we have investigated the turbulent behavior of equation (1) from different standpoints. Several different concepts have been introduced and developed. We would like to stress several points. First, some of the concepts introduced in the study of hydrodynamic turbulence lead to a fairly good understanding of the energy spectrum of turbulent solutions of Eq. (1). Secondly, the results obtained concerning the linear growth of the number of positive LN with size L answer—at least numerically—some open theoretical questions concerning the problem of turbulent degrees of freedom.

Finally we have proposed explanations for the observed spectra of LV for Eq. (1). Checking these concepts and extending them to three-dimensional Navier–Stokes equations will certainly lead to a better understanding of turbulence.

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